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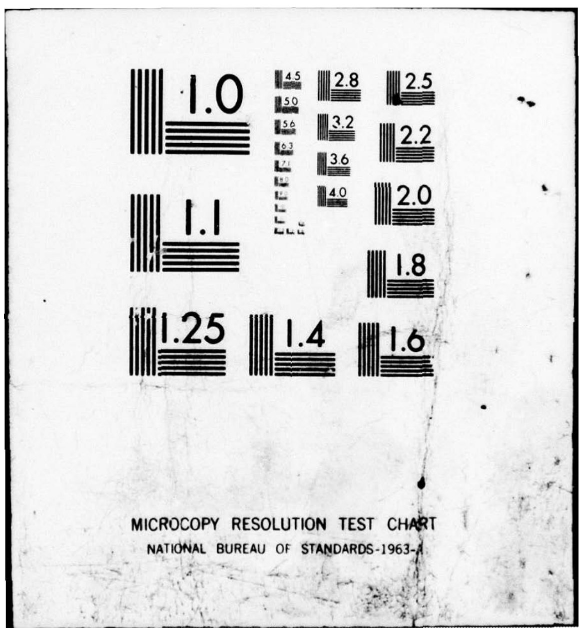
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SPIKE SWAPPING IN BASIS REINVERSION

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and
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Comments and criticisms from interested readers are cordially invited.

ABSTRACT

P(3) P(4)

An important component of a large scale linear programming system is the reinversion routine. This paper addresses an important ancillary technique for implementing a reinversion routine utilizing the well-known P^3 and P^4 pivot agenda algorithms of Hellerman and Rarick. Production of factors during reinversion typically involves a left-to-right pivoting process. Unfortunately, during the left-to-right process, a proposed pivot element of a spike column may be zero, in which case columns are interchanged in an attempt to obtain a pivotable column. In this paper we prove that the only columns which need be considered for the interchange with a nonpivotable spike *are other spikes lying to the right within the same external bump.*

I. INTRODUCTION

Production linear programming codes maintain the inverse of the basis in a factored form as the product of elementary column matrices (i.e. matrices which differ from an identity in one column) or elementary row matrices (i.e. matrices which differ from an identity in one row) or some combination of the two. The vector (either the column or the row) which distinguishes these matrices from an identity matrix are known as *eta vectors* and the sequence of these vectors is known as the ETA file. Each basis change results in appending at least one eta vector to the ETA file. Since both the time per pivot and numerical error increase as the length of the ETA file increases, it becomes necessary to periodically obtain a new factorization of the basis inverse. For an m column basis, this is accomplished in m steps by beginning with an identity and successively pivoting into the identity, columns of the basis. At the termination of the process the ETA file will represent the inverse of the current basis. If in this process a proposed pivot element is zero, then columns of the basis must be interchanged until a nonzero pivot element can be found. Although reinversion techniques have been discussed in the literature (e.g. [3, 4, 5]) a strategy for swapping columns has never been explicitly detailed. The objective of this paper is to present new results which address the problem of column swapping. We prove that the choices for the column interchanges can be limited to a few special columns.

II. PRODUCT FORM OF THE INVERSE

Let B be any m x m nonsingular matrix. One of the most common factorizations for B^{-1} is the product form which corresponds to the method for solving a system of linear equations known as Gauss-Jordan reduction (see Section 5.2 of [1]). This procedure is used to represent B^{-1} as the product of matrices each of the form

$$Z = \left[\begin{array}{c|c|c} I & \begin{matrix} z_1 \\ \vdots \\ z_{j-1} \end{matrix} & \\ \hline & z_j & \\ \hline & \begin{matrix} z_{j+1} \\ \vdots \\ z_m \end{matrix} & I \end{array}\right] . \qquad (1)$$

A few observations concerning Z are obvious.

<u>Proposition 1.</u>

Z is nonsingular if and only if $z_j \neq 0$.

<u>Proposition 2.</u>

Let β be any m-component vector having $\beta_j = 0$. Then $Z\beta = \beta$.

<u>Proposition 3.</u>

Let β be any m-component vector having $\beta_j \neq 0$, and let e^j denote the vector having j^{th} component 1 and all other components zero.

$$\text{Let } z_k = \left\{ \begin{array}{l} -\beta_k/\beta_j, \text{ if } k \neq j \\ 1/\beta_j, \text{ if } k = j \end{array} \right\} . \qquad \text{Then } Z\beta = e^j.$$

We now give the product form algorithm for B^{-1}. For this presentation we adopt the notation that A(i) denotes the i^{th} column of the matrix A.

ALG 1: PRODUCT FORM FACTORIZATION

0. *Initialization*

Interchange columns of B, if necessary, so that the first component of B(1) is nonzero. Set $i \leftarrow 1$, $\beta \leftarrow B(1)$, and go to 3.

1. *Update Column*

Set $\beta \leftarrow E^{i-1} \cdots E^{1}B(i)$.

2. *Swap Columns If Pivot Element Equals Zero*

If $\beta_i \neq 0$, go to 3; otherwise, there is some column B(j) with $j > i$ such that the i^{th} component of $E^{i-1} \cdots E^{1}B(j)$ is nonzero. Interchange B(j) and B(i) and return to 1.

3. *Obtain New Column Eta*

Set $E^{i}(k) \leftarrow e^{k}$, for all $k \neq i$, and

$$\text{set } E^{i}_{ki} \leftarrow \begin{cases} 1/\beta_i, & \text{for } k = i \\ -\beta_k/\beta_i, & \text{otherwise.} \end{cases}$$

4. *Test for Termination*

If $i = m$, terminate; otherwise, $i \leftarrow i + 1$ and go to 1.

At the termination of ALG 1, $B^{-1} = E^{m} \cdots E^{1}$, and each factor $E^{1}, \ldots, E^{m}$, take the form of Z in (1).

In the following two propositions we show that if in step 2, $\beta_i = 0$, then the proposed interchange is always possible. Consider the following:

Proposition 4.

For $i \leq j$, $E^j \cdots E^1 B(i) = e^i$.

Proof. By the construction of E^i and Proposition 3, $E^i \cdots E^1 B(i) = e^i$. By Proposition 2, $E^j \cdots E^{i+1} e^i = e^i$. So $E^j \cdots E^1 B(i) = e^i$.

Using Proposition 4 we may now show the following:

Proposition 5.

For $2 \leq i \leq m$, let $\beta = E^{i-1} \cdots E^1 B(i)$. If $\beta_i = 0$, there is some $j > i$ such that $[E^{i-1} \cdots E^1 B(j)]_i \neq 0$.

Proof. Suppose $[E^{i-1} \cdots E^1 B(j)]_i = 0$ for all $j > i$. By the construction of $E^1, \ldots E^{i-1}$, in ALG 1, and Proposition 1, each factor is nonsingular. Since B is nonsingular, $E^{i-1} \ldots E^1 B$ is nonsingular. By Proposition 4, $E^{i-1} \ldots E^1 B(j) = e^j$ for $1 \leq j \leq i - 1$. Hence, the i^{th} row of $E^{i-1} \ldots E^1 B$ is all zero, a contradiction.

III. BUMP AND SPIKE STRUCTURE

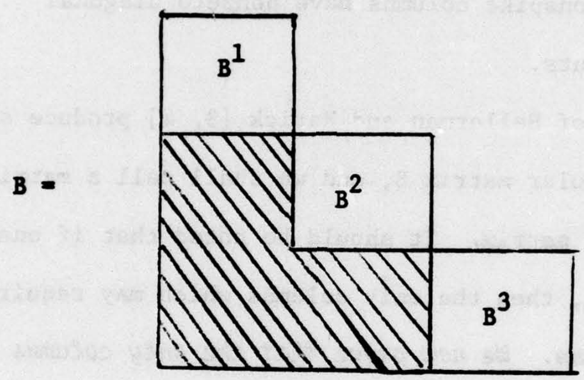
In order to minimize the core storage required to represent the ETA file, i.e. $E^1, \ldots, E^m$, the rows and columns of B are interchanged in an attempt to place B in lower triangular form. If this can be accomplished, then the m nonidentity columns of $E^1, \ldots, E^m$, have the same sparsity structure as B. Consider the following proposition:

Proposition 6.

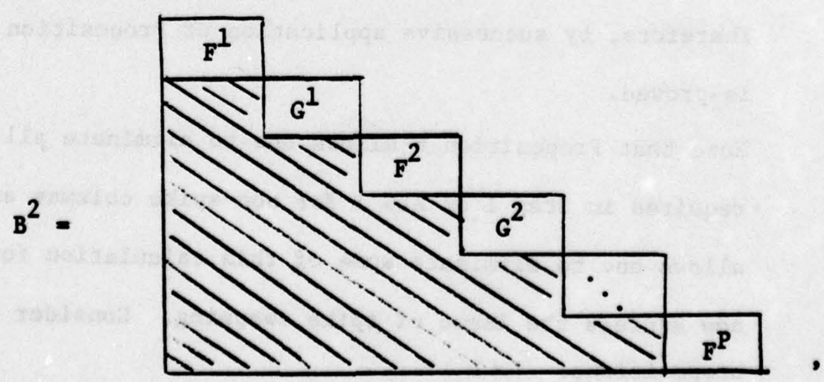
If the first j-1 components of B(j) are zero for $j > 2$, then $E^{j-1} \ldots E^1 B(j) = B(j)$.

Proof. This follows directly from successive application of Proposition 2. Therefore, if B is lower triangular, the factored representation of B^{-1} may be stored in approximately the same amount of core storage as B itself. In practice it is unnecessary to calculate the elements $1/\beta_k$ and $-\beta_i/\beta_k$ in Step 3 of ALG 1. It sufficies to store k and the elements of β_i. It may prove advantageous to store $1/\beta_k$, in addition. If Proposition 6 applies for B(k), then β = B(k) and the only additional storage required is for the index k (and possibly $1/\beta_k$). Clearly, this results in substantial core storage savings compared to storing B^{-1} explicitly.

If B cannot be placed in lower triangular form, then it is placed in the form:



where B^1 and B^3 are lower triangular matrices with nonzeroes on their diagonals. We assume that if B^2 is nonvacuous, every row and column has at least two nonzero entries, so that no rearrangement of B^2 can expand the size of B^1 or B^3. B^2 is called the *bump section*, the *merit section* or the *heart section*. We further require the heart section to assume the following form:



where G^k's are either vacuous or lower triangular with nonzeroes on the diagonal. The only partitions in B having columns with nonzeroes above the diagonal are the F^k's which are called *external bumps*. The columns extending above the diagonal are called *spikes* or *spike columns*. An external bump is characterized as follows:

(i) the last column of an external bump will be a spike with a nonzero lying in the topmost row of the external bump, and

(ii) the nonspike columns have nonzero diagonal elements.

The algorithms of Hellerman and Rarick [3, 4] produce such a structure for any nonsingular matrix B, and we shall call a matrix having this structure an HR matrix. It should be noted that if one applies ALG 1 to an HR matrix, then the only columns which may require an interchange are spike columns. *We now prove that the only columns which need be considered for this interchange are other spikes in the same external bump.*

Consider the following result:

Proposition 7.

Let B(i) with $i \geq 2$ correspond to the first column of some external bump, F^k, and let B(j) be a spike in F^k. Then $E^{i-1} \ldots E^1 B(j) = B(j)$.

Proof. Note that the first $i - 1$ components of B(j) are zero. Therefore, by successive application of Proposition 2, the result is proved.

Note that Proposition 6 allows one to eliminate all of the calculation required in Step 1 of ALG 1 for non-spike columns and Proposition 7 allows one to eliminate some of this calculation for spikes. We now address the issue of spike swapping. Consider the following propositions:

Proposition 8.

Any spike $B(j)$ which is not pivotable cannot be interchanged with a spike $B(k)$, $k > j$, from another external bump, to yield a pivotable column.

Proof. Since $B(k)$ is from an external bump lying to the right of the external bump containing $B(j)$, $B_j(k) = 0$. By repeated application of Proposition 2, $E^{j-1} \ldots E^1 B(k) = B(k)$. Thus $B(j)$ cannot be interchanged with $B(k)$ to yield a pivotable column.

Proposition 9.

Any spike $B(j)$ which is not pivotable cannot be interchanged with a non-spike column $B(k)$, $k > j$, to yield a pivotable column.

Proof. Let $B(k)$, with $k > j$ correspond to any non-spike column. From Proposition 6, $E^{j-1} \ldots E^1 B(k) = B(k)$. Since the j^{th} component of $B(k)$ is zero, $B(j)$ cannot be interchanged with $B(k)$, to yield a pivotable column.

Proposition 10.

Any spike column $B(j)$, which is not pivotable can be interchanged with a spike, $B(k)$, with $k > j$ within the same external bump, to yield a pivotable column.

Proof. If $B(j)$ is not pivotable, then by Proposition 5 there exists a column $B(k)$ with $k > j$ which is pivotable. By Proposition 8, $B(k)$ cannot be a spike from a different external bump. By Proposition 9, $B(k)$ cannot be a non-spike. Hence $B(k)$ must be a spike from the same external bump.

We now revise ALG 1 incorporating the results of Propositions 6, 7, and 10.

ALG 2: PRODUCT FORM FACTORIZATION

FOR A HR MATRIX

0. *Initialization*

   (same as ALG 1)

1. *Update Spike*

   Let B(k) correspond to the first column of the external bump containing B(i). Set $\beta \leftarrow E^{i-1} \cdots E^{k}B(i)$.

2. *Swap Spikes If Pivot Element Equals Zero*

   If $\beta_i \neq 0$, go to 3; otherwise, there is some spike B(j) in the same external bump having $j > i$ such that the i^{th} component of $E^{i-1} \cdots E^{k}B(j)$ is nonzero. Interchange B(j) and B(i) and return to 1.

3. *Obtain New Column Eta*

   (same as ALG 1)

4. *Test for Termination*

   If $i = m$, terminate; otherwise, $i \leftarrow i + 1$.

5. *Test for Spike*

   If B(i) is a spike, go to 1; otherwise, set $\beta \leftarrow B(i)$ and go to 3.

The results of this section also hold if one uses the elimination form of the inverse, which corresponds to the Gauss reduction. The difference between the product and the elimination form factorizations is that for the latter one develops $2m - 1$ triangular factors such that $B^{-1} = E^{2m-1} \cdots E^{1}$. A factor of the form shown in (1) is called *triangular* if either $z_1 = \ldots = z_{j-1} = 0$ or $z_{j+1} = \ldots = z_m = 0$. The details of the elimination form algorithm are given in [2].

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An important component of a large scale linear programming system is the reinversion routine. This paper addresses an important ancillary technique for implementing a reinversion routine utilizing the well-known P^3 and P^4 pivot agenda algorithms of Hellerman and Rarick. Production of factors during reinversion typically involves a left-to-right pivoting process. Unfortunately, during the left-to-right process, a proposed pivot element of a spike column may be zero, in which case columns are interchanged in an attempt to obtain a

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